ANNEX XX

Combined Continuous Time Random Walk in Position and Velocity Space: Derivation of the Basic Equations

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INTRODUCTION

Diffusion in confined plasmas is known to be anomalous [5]. Classical models for diffusion (e.g. the Fokker-Planck equation) are not able to capture the phenomenon of anomalous diffusion, so that there is a need for non-classical diffusion models. Models able to deal with anomalous diffusion are e.g. fractional or non-linear diffusion equations, and Continuous Time Random Walk (CTRW; see e.g. [1]). A different, complementary approach to the understanding of anomalous diffusion is the modeling of plasma turbulence (see e.g. [2]), in combination with test particle simulations (see e.g. [3]), which allow to determine some of the probabilistic laws that enter the CTRW formalism.

Here, we develop the probabilistic CTRW equations as appropriate to confined plasma in a turbulent state. The CTRW framework has been used so-far to treat position space alone, in general as well as in applications to confined plasmas (see e.g. [4]). New in our approach is that we also take momentum space into account and study the combined random walk in momentum and position space, i.e. we construct a more realistic model for diffusion that also is able to deal with heating and particle acceleration, which are both phenomena seen only in momentum space.

THE CTRW AND ITS EQUATIONS

The random walk we consider here consists in three basic steps (see Fig. 1):

A particle is accelerated/decelerated in a localized spatial region, changing its momentum $\vec{p}$ by the amount $\Delta\vec{p}$,

- thereafter, the particle is trapped for a time t_w,
- and finally, it performs a free jump $\Delta\vec{r}$ in position space.

When the basic cycle is completed, a new cycle immediately starts. The acceleration times and the free flight times are neglected (or we can consider them to be incorporated in the waiting times; the formalism presented here actually allows both of them to be taken into account, at the expense of having more complicated equations).

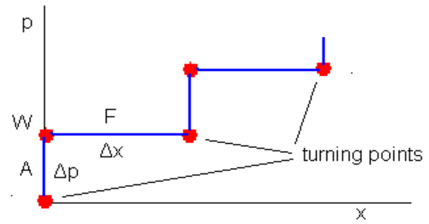

Figure 1. Sketch of the CTRW in position (x) and momentum (p) space. A particle starts at a turning point, undergoes an acceleration event (A), enters a trapping event (W), performs a free flight (F), and arrives at a new turning point to start a new cycle.

From the random walk point of view, the basic quantity is the probability density function (pdf) $\psi(\Delta\vec{p},\Delta\vec{r},t_w)$ of the walk increments, which yields the probability that a particle gets an increment in momentum $\Delta\vec{p}$, it is trapped and waits for a time t_w, and it makes a free jump $\Delta\vec{r}$ in position space. $\Delta\vec{p}$, t_w, and $\Delta\vec{r}$ are all independent random variables. The joint pdf ψ decouples thus as

$$\psi(\Delta\vec{p},\Delta\vec{r},t_w) = q_{\Delta\vec{p}}(\Delta\vec{p})q_{\Delta\vec{r}}(\Delta\vec{r})q_w(t_w) \tag{1}$$

where $q_{\Delta\vec{p}}(\Delta\vec{p})$, $q_{\Delta\vec{r}}(\Delta\vec{r})$, and $q_w(t_w)$ are the pdf's for the different increments.

The pdf of the turning-points: As turning points of the random walk we define the points in $\vec{r}$ - $\vec{p}$ -space where the particles arrive at and will immediately undergo an acceleration event (see Fig. 1). Two turning-points are thus separated by an acceleration event, a trapping event, and a free jump in position-space. The rate Q of particles just arriving at a turning point at $(\vec{r},\vec{p})$ is recursively given as

$$Q(\vec{r},\vec{p},t) = \int d^3\Delta r \int d^3\Delta p \int_0^t dt_w Q(\vec{r}-\Delta\vec{r},\vec{p}-\Delta\vec{p},t-t_w)q_{\Delta\vec{r}}(\Delta\vec{r})q_{\Delta\vec{p}}(\Delta\vec{p})q_w(t_w)$$
$$+\delta(t)P(\vec{r},\vec{p},t=0) \tag{2}$$

The first term on the right hand side just describes a completed cycle of a CTRW step in position space, momentum space, and time. The second term on the right hand side takes the initial conditions into account, $P(\vec{r},\vec{p},t=0)$ is the particle distribution at time $t=0$.

The propagator: The propagator $P(\vec{r},\vec{p},t)$, which is a pdf, gives the probability with which a particle is at position $(\vec{r},\vec{p})$ at time t. It is determined through

$$P(\vec{r},\vec{p},t)=\int d\Delta\vec{p}\,'\int_o^t dt'_w\, Q(\vec{r},\vec{p}-\Delta\vec{p}\,',t-t'_w)q_{\Delta\vec{p}}(\Delta\vec{p})\Phi_w(t) \qquad (3)$$

where Φ_w is the probability for a particle to be for a time t'_w in an trapping event, which itself lasts longer than t'_w,

$$\Phi(t'_w)=\int_{t''_w\ge t'_w} dt''_w\, q_w(t''_w) \qquad (4)$$

Eq. (3) states that a particle is at position $(\vec{r},\vec{p})$ at time t if it was at a turning point $(\vec{r},\vec{p}-\Delta\vec{p}\,')$ at time $t-t'_w$, it underwent an acceleration event which increased its momentum by $\Delta\vec{p}\,'$, and it is now in a trapping event whose duration is at least t'_w. Note that no time is consumed in the acceleration and free flight events, so that we cannot locate the particles during these two kinds of events.

Reformulating: Inserting Eq. (2) into Eq. (3) and changing the order of integration, we can eliminate Q and formulate the evolution equation for P completely in terms of P

$$P(\vec{r},\vec{p},t)=\int d^3\Delta r\int d^3\Delta p\int_0^t dt_w P(\vec{r}-\Delta\vec{r},\vec{p}-\Delta\vec{p},t-t_w)q_{\Delta\vec{r}}(\Delta\vec{r})q_{\Delta\vec{p}}(\Delta\vec{p})q_w(t_w)$$
$$+\int d^3\Delta p'\,P(\vec{r},\vec{p}-\Delta\vec{p}\,',0)q_{\Delta\vec{p}}(\Delta\vec{p}\,')\Phi_w(t) \qquad (5)$$

The propagator contains all the statistical information about the random walk, including the mean square displacement in position as well as in velocity, etc,

CONCLUSION

We have derived self-consistent equations for the combined random walk in position and momentum space. In the version presented here, we neglected the free flight times and the acceleration times. From the form of the equations, it can be expected that asymptotic solutions can be determined analytically. Monte Carlo simulations have though shown that small times and the low to intermediate energy regime show interesting behavior (effect of just few acceleration events, heating) that cannot be captured by asymptotic solutions. Our main focus will thus be on the numerical solution of the CTRW integral equation, which will yield the solution also at small times and for all the energy range. In fact it will be possible to discuss the relevance of asymptotic solutions to the problem of anomalous transport. We currently work on the numerical solution of the CTRW equation, applying the Nystrom method. Once numerical solutions will be available, they will be compared to Monte Carlo simulations as a test. Most interesting will then be to identify the changes and new features in the CTRW that come in through the inclusion of momentum space. Finally, we plan to combine our CTRW approach with our simulations of plasma turbulence [2] and with Self-Organized Criticality models in order to have realistic assumptions on the basic free parameters of the CTRW, the pdf 's of the increments, which will allow to analyze anomalous transport in confined, turbulent plasmas regarding both, position and momentum space.

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