ANNEX XIII

Higher-order diffusion equation for resonant wave-particle interactions

Y. Kominis[1], K. Hizanidis[1], A. K. Ram[2], and A. Anastasiadis[1,3]
[1]School of Electrical and Computer Engineering, National Technical University of Athens, Athens, Greece
[2]Plasma Science and Fusion Center, Massachusetts Institute of Technology, Cambridge, Massachusetts, USA
[3]Institute for Space Applications and Remote Sensing, National Observatory of Athens, Athens, Greece

INTRODUCTION AND MODEL

Wave-particle interactions, occurring in the context of ECRH and ECCD, have been shown to result in diffusion-like processes in the velocity (or action) space, for cases where the underlying Hamiltonian system is non-integrable. In such cases, under the assumption of small perturbation of the Hamiltonian (near-integrable system) the action diffusion has been shown to be described by a quasilinear Fokker-Planck equation of the form

$$\frac{\partial F}{\partial t} = \frac{\partial}{\partial J}\left(D(J,t)\frac{\partial F}{\partial J} \right), \tag{1}$$

where J is the action, $F(J)$ is the action distribution function and $D(J,t) = \lim_{\Delta t \to 0} < \left(\Delta J\right)^2 >_\theta /(2\Delta t)$ is the quasilinear diffusion coefficient. However, cases of stronger perturbation are not within the domain of the validity of the quasilinear theory. Thus, it is necessary to generalise the diffusion equation as well as its derivation method, by utilising a rigorous perturbation method, which provides the quasilinear diffusion equation (1), at lowest order, but also includes higher order terms. In this work, we utilise the Lie transform perturbation theory [1] in order to derive a higher order diffusion equation for the simplified model

$$H = J^2 - \frac{1}{2}E(2J)^{k_0/2} e^{ik_0\theta} g(t), \tag{2}$$

describing the weakly-relativistic motion of a particle interacting resonantly with an electromagnetic wave whose frequency is close to the k_0-th harmonic of the cyclotron frequency. The action-angle variables of the unperturbed system (J,θ) are the transverse kinetic energy and the rotation angle of the particle with respect to the rotating electric field, while E and $g(t)$ are the field amplitude and profile, respectively.

HIGHER-ORDER DIFFUSION EQUATION

In the context of Lie transform perturbation theory [1] the evolution of the distribution function in an infinitesimal time interval $[t, t+\Delta t]$ is

$$f(J,\theta)_{t+\Delta t} - f(J,\theta)_t = \left(T^{-1} - I\right) f(J,\theta)_t, \tag{3}$$

where T^{-1} is the inverse Lie transform defined and calculated as in [1]. By considering the limit $\Delta t \to 0$, we obtain

$$\frac{\partial F(J,t)}{\partial t} = \frac{\partial \left\langle \left(T^{-1} - I\right) \right\rangle_\theta}{\partial t} F(J,t), \tag{4}$$

and by utilising eq. (12) of [1], we finally have the higher-order diffusion equation

$$\begin{aligned}
\frac{\partial F}{\partial t} = {} & k_0^2 \frac{\partial}{\partial J}\left[\left(| F_{1,1} |^2 \right)_t \frac{\partial F}{\partial J} \right] + k_0^2 \frac{\partial}{\partial J}\left[\left(| F_{2,2} |^2 \right)_t \frac{\partial F}{\partial J} \right] + \frac{k_0^2}{3}\frac{\partial}{\partial J}\left[\Re\left(\bar{F}_{1,1} F_{3,1} \right) \frac{\partial F}{\partial J} \right] \\
& - \frac{k_0^3}{6}\frac{\partial}{\partial J}\left[4\Im\left(F_{1,1}^2 \bar{F}_{2,2} \right)_t \frac{\partial^2 F}{\partial J^2} + 2\frac{\partial}{\partial J}\left(\Im\left(F_{1,1}^2 \bar{F}_{2,2} \right) \frac{\partial F}{\partial J} \right) \right] \\
& + \frac{k_0^4}{12}\frac{\partial}{\partial J}\left[\left(3\frac{\partial^2}{\partial J^2}\left(| F_{1,1} |^2 \frac{\partial F}{\partial J} \right) - \frac{\partial^2 \left(| F_{1,1} |^2 \right)}{\partial J^2}\frac{\partial F}{\partial J} \right) | F_{1,1} |^2 \right]_t
\end{aligned}$$

which is valid up to fourth order with respect to the perturbation strength. It is worth mentioning that the first term of this equation coincides with the quasilinear Fokker-Planck diffusion equation (1).

REFERENCES

[1] Y. Kominis, K. Hizanidis, O. Dumbrajs, K. A. Avramides and J. L. Vomvoridis, *Annex VII* in *Fusion Project, Association EURATOM-Hellenic Republic, Progress Report 2006.*