ANNEX 14

Analytical and numerical study of magnetohydrodynamic natural convection in an internally heated horizontal shallow cavity

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INTRODUCTION

A study is presented of the two-dimensional MHD natural convection of an electrically conductive fluid in an internally heated horizontal shallow cavity in the presence of an external vertical magnetic field. All walls are electrically insulated, with the horizontal being adiabatic and the vertical isothermal. The method of the matched asymptotic expansions is used to obtain solutions of the flow and heat transfer characteristics. This analysis, valid for large cavity aspect ratios and for any magnetic field strength, is particularly helpful for the inexpensive determination of the flow field. In addition to the analytical solutions, the same flow is also determined numerically for a range of Hartmann, Prandtl and Rayleigh numbers in order to verify the accuracy and validity of the analytical results and to calculate the constants arising by the analytical approach. The main feature of the present flow is a symmetric double-cell Hadley circulation with the fluid ascending in the hotter centre of the cavity and descending near the vertical cold walls. The comparison of the results of the analytical and numerical solutions was found to be fairly good indicating the correctness of the analysis and its applicability.

MATHEMATICAL FORMULATION

Consider the horizontal two-dimensional rectangular shallow cavity of Fig. 1 of large aspect ratio *L/h* (length/height) filled with an internally-heated electrically conductive fluid. The flow is assumed to be laminar, steady and two-dimensional while Joule heating and viscous dissipation effects are neglected. Also, the quasi-static (or low magnetic Reynolds number, R_m) approximation is employed for the fluid magnetic induction-momentum connection because the induced magnetic field is considered negligible compared to the external magnetic field B_0.

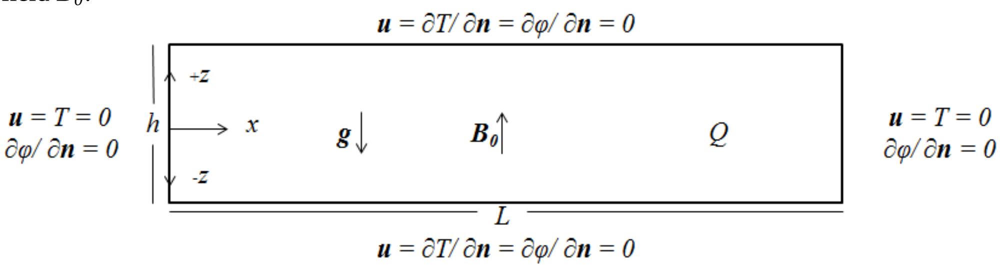

Fig. 1: Geometry considered and boundary conditions

Based on the above assumptions and using the Oberbeck-Boussinesq approximation for the fluid density thermal variations, the dimensionless governing equations of the present 2-D steady incompressible MHD flow are as follows:

$$\nabla^4\psi = \mathrm{Pr}^{-1}\frac{\partial(\nabla^2\psi,\psi)}{\partial(x,z)} + Ha^2\frac{\partial^2\psi}{\partial z^2} + Ra\frac{\partial \mathrm{T}}{\partial x} \qquad (1)$$

$$\nabla^2 T + 1 = \frac{\partial(T,\psi)}{\partial(x,z)} \qquad (2)$$

Where $\Psi=\psi/\alpha$ is the dimensionless streamfunction defined via the relationships $u=\partial\psi/\partial z$ and $w=-\partial\psi/\partial x$, $\alpha=k/\rho c_p$ is the fluid thermal diffusivity, $X=x/h$ and $Z=z/h$ are the dimensionless *x* and *z* coordinates, respectively, and $\Theta=T\rho c_p a/h^2 Q$ is the dimensionless fluid temperature (note that Ψ is changed to ψ, X to x, Z to z and Θ to T thereon). Furthermore, $Pr=\nu/\alpha$ is the fluid Prandtl number, $Ra=g\beta Qh^5/\rho c_p \nu\alpha^2$ the Rayleigh number and $Ha=B_0 h(\sigma/\rho\nu)^{1/2}$ the Hartmann number. Finally, the notation $\partial(i,j)/\partial(x,z)$ stands for $(\partial i/\partial x)(\partial j/\partial z)$ - $(\partial j/\partial x)(\partial i/\partial z)$.

The method of the matched asymptotic expansions is used to obtain the basic flow and temperature fields from Eqs. (1) and (2) and their boundary conditions, as shown in Fig. 1. Thus, the core solutions for the temperature, streamfunction and vertical velocity read as:

$$\theta_0 = a_m^{-1} R_s^{-2}[cosh(\frac{2}{3} y_m) - \frac{1}{2} cosh(\frac{4}{3} y_m) - cosh(\frac{2}{3} y_{m,0}) + \frac{1}{2} cosh(\frac{4}{3} y_{m,0})] \tag{3}$$

$$\psi_0 = -2 \frac{a_m^{-1/2} f(z)}{Ha^2} sinh(\frac{1}{3} y_m) \tag{4}$$

$$w_{an} = -\frac{\partial \psi_0}{\partial \xi} = \frac{R_s f(z)}{Ha^2[1 + 4 sinh^2(\frac{1}{3} y_m)]} \tag{5}$$

where R_s=RaL is the scaled Rayleigh number and y_m, $y_{m,0}$, a_m, f are functions of Ha, R_s, z and ξ.

Regarding heat transfer, an average Nusselt number for the cavity based on the heat transfer through the side wall at x=0 relative to the maximum temperature difference in the cavity, located at (ξ=1/2, z=1/2), can be estimated from Daniels and Jones [1]:

$$Nu_{av} = \frac{1}{T\left(L/2, 1/2\right)} \int_{-1/2}^{1/2} \frac{\partial T}{\partial \xi}(0,z)dz = \frac{1}{2L\theta_0\left(1/2\right)}, \quad for \ L \to \infty \tag{6}$$

The present numerical model was based on the in-house model developed by Iatridis [2] which used the open source CFD library “OpenFOAM” [3]. Furthermore, it was initially validated by successful comparison with the results of Al-Najem et al. [4] and Ozoe et al. [5]. Thus, the numerical simulations solved the following system of dimensionless flow-governing equations:

$$\nabla \cdot \boldsymbol{u} = 0 \tag{7}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + \left(\boldsymbol{u} \cdot \nabla\right)\boldsymbol{u} = -\nabla p + Pr\nabla^2 \boldsymbol{u} + RaPrT + Ha^2 Pr\boldsymbol{J} \times \boldsymbol{B_0} \tag{8}$$

$$\frac{\partial T}{\partial t} + \left(\boldsymbol{u} \cdot \nabla\right)T = \nabla^2 T + 1 \tag{9}$$

$$\boldsymbol{J} = -\nabla\varphi + \boldsymbol{u} \times \boldsymbol{B}_0 \tag{10}$$

$$\nabla^2\varphi = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}_0) \tag{11}$$

The equations were made dimensionless by scaling the length with the cavity height h, velocity (u) with a/h, pressure (p) with $\rho\alpha^2/h^2$, magnetic field (B) with B_0, current density (J) with $\sigma B_0 \alpha/h$, electrostatic potential (φ) with B_0/α and time (t) with h^2/α. Moreover, the dimensionless temperature is defined as $T=(T-T_0)/(Qh^2/k)$.

Equations (7) to (11) together with their boundary conditions were solved by using a finite volume method based on the transient pressure-velocity coupling algorithm PISO [6]. A Crank-Nicolson scheme was used for the transient terms, central differences for the Laplacian and pressure terms and a hybrid difference scheme for the convection terms. At each step, the solution is iterated until the residuals of the mass, momentum, electrostatic potential and temperature equations became smaller than 10^{-7}. A non-uniform staggered grid of (400x80) lines in the horizontal and vertical directions, respectively, with finer distribution near the walls was tested and considered adequate for the present study. Special attention for the distribution of grid lines was given because the Hartmann boundary layers are narrow and must be adequately covered by the numerical grid.

RESULTS AND DISCUSSION

The resulting flow consists of a symmetric double-cell Hadley circulation with the fluid ascending in the hotter center of the cavity and descending near the cold vertical walls. Both analytical and numerical results demonstrate that the Hartmann (Ha) and the scaled Rayleigh (R_s) numbers can influence notably the heat transfer mechanism. More specifically, the fluid is decelerated by the external magnetic field leading to the dominance of heat conduction and reducing the heat transfer. The same occurs when R_s decreases, since the reduction of internal heating reduces the temperature gradients in the cavity and, thus, the effects of buoyancy. Consequently, the fluid temperature is kept high and the vertical walls lose their ability to cool the interior fluid. Fig. 2 shows that as Ha decreases the core temperature also decreases with the maximum value located at mid-cavity. Fig. 3 indicates that as R_s increases (or Ha decreases) the natural convection is intensified and, thus, the average Nusselt number relative to the maximum temperature difference in the cavity (Nu_{av}) increases. In addition, the analytical vertical velocity in the core region cannot attain the negative values near the vertical walls which appear in the numerical results indicating a downward fluid motion in this region. Figs. 4 and 5 show again that the natural convection becomes stronger as R_s increases (or Ha decreases), as indicated by the increased velocity and streamfunction values. The analytical values of streamfunction were approaching the numerical ones near the walls as the fluid motion was decelerated. The comparison between the numerical and analytical results showed that the latter are valid in the core region for low internal heating and strong magnetic fields. Finally, instabilities are expected to arise at sufficiently large Rayleigh numbers.

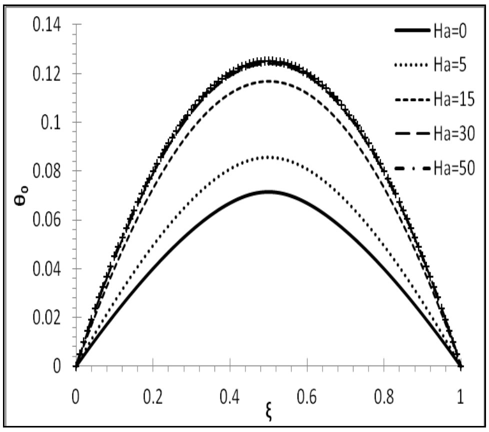

Fig.2: Analytical core temperature profiles at mid-cavity height for R_s=3000 and various Ha.

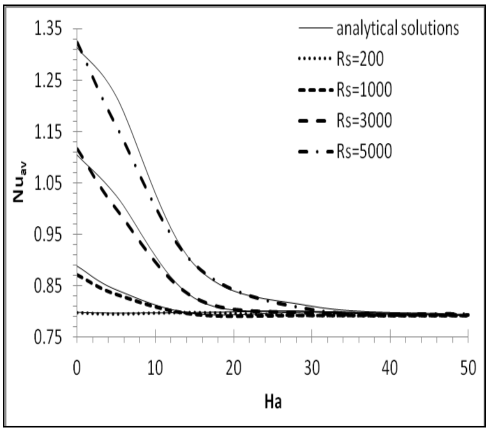

Fig.3: Variation of the average Nusselt number with Ha for various values of R_s.

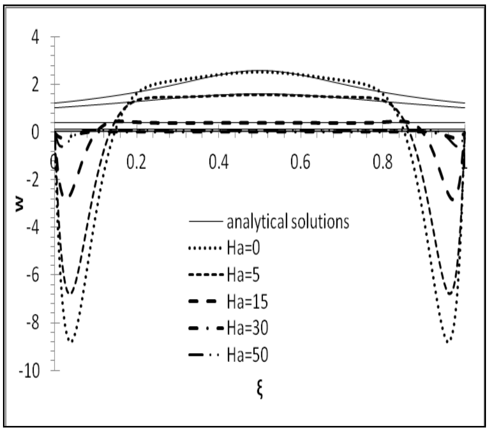

Fig.4: Vertical velocity profiles at mid-cavity height for various values of Ha and R_s=1000.

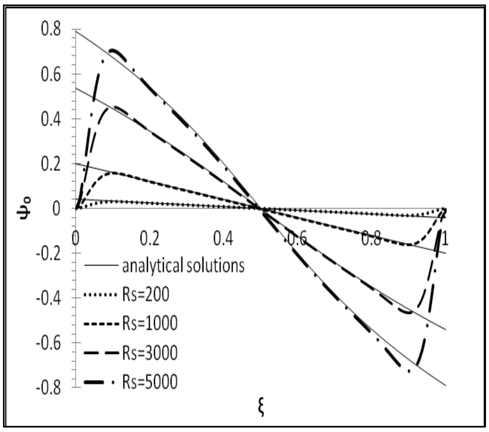

Fig.5: Distribution of core streamfunction at mid-cavity height for various values of R_s and Ha=15.

REFERENCES

[1] P. Daniels and O. Jones, "Convection in a shallow cavity due to internal heat generation", Int. J. Heat & Mass Transfer **41**, 3979 (1998).

[2] A. J. Iatridis, "Magnetohydrodynamic flow in fusion machine ducts: Transition from laminar to turbulent flow", PhD Thesis, Univ. Thessaly, Greece (2012).

[3] H. Weller, H. Jasak and C. Fureby, "A tensorial approach to computational continuum mechanics using object-oriented techniques", Computers in Physics **12**, 620 (1998).

[4] N. Al-Najem, K. Khanafer and M. El-Rafaee, "Numerical Study of laminar natural convection in tilted enclosure with transverse magnetic field", Int. J. Numerical Methods in Heat & Fluid Flow **8**, 651 (1998).

[5] H. Ozoe and K. Okada, "The effect of the direction of the external magnetic field on the three- dimensional natural convection in a cubical enclosure", Int. J. Heat & Mass Transfer **32**, 1939 (1989).

[6] R. Issa, "Solution of the implicitly discretised fluid flow equations by operator-splitting", J. Computational Physics **62**, 40 (1986).