ANNEX 15

Finite Element Methodology with Adaptive Mesh Refinement for ODEs with Boundary Layers

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INTRODUCTION

A linear second order ODE containing a small parameter multiplying the second derivative, thus generating a boundary layer near one of the boundaries, was solved via the standard finite element methodology using Lagrangian basis functions. The exact error was obtained upon comparing against the exact solution. Next, an a-posteriori error estimate was obtained based on the symmetric part of the operator and performing an element-wise calculation using the residual method with flux equilibration and a higher order finite element description. The mesh was subsequently adapted in regions of the domain where the error estimate acquired values beyond a certain predetermined threshold with respect to the numerical solution. Increased convergence rates were compared and the error estimates were compared with the actual numerical error.

FORMULATION AND NUMERICAL SOLUTION

An effort is made to implement a numerical technique that captures dissipative layers in MHD applications with minimal mesh requirements. This methodology can be extended to capture Hartmann and side layers in non-ideal MHD [1] or critical layers in plasma modeling [2]. The basic concept follows from earlier ideas based on a posteriori error estimates using element residual methods [3]. To this end the following ODE is solved using the finite element methodology :

$$-\varepsilon y'' + 2xy' + 2y = 0, \quad y(x=0) = e^{-1/\varepsilon} \quad y'(x=1) = 2/\varepsilon \tag{1}$$

The exact solution of the above ODE is $y_{Ex} = e^{\frac{x^2-1}{\varepsilon}}$ which exhibits a boundary layer at x=0. The numerical solution is obtained in terms of standard finite element analysis using a uniform mesh as a starting point :

$$\hat{y}(x) = \sum_{i=1}^{N+1} a_i B_i(x) \quad D(\hat{y}, B_i) \equiv \int_\Omega \left(\varepsilon \frac{d\hat{y}}{dx}\frac{dB_i}{dx} + B_i 2x \frac{d\hat{y}}{dx} + 2B_i \hat{y} \right) dx = B_i \varepsilon \frac{d\hat{y}}{dx}\bigg|_0^1$$
$$\rightarrow \sum_{i=1}^{N=1} \left[\int_0^1 \left(\frac{dB_i}{dx} \varepsilon \frac{dB_j}{dx} + 2xB_i \frac{dB_j}{dx} + 2B_i B_j \right) dx \right] a_j = B_i \varepsilon \frac{d\hat{y}}{dx}\bigg|_0^1 \tag{2}$$

with $B_i(x)$ the Lagrangian basis functions. As a next step error estimate function φ is calculated on the element level and through it an error estimation is obtained, following [3,4], using the norm consisting of the symmetric part of the ODE:

$$\|\in\|_{h,K} \equiv \|y_{ex} - \hat{y}\|_{h,K}$$
$$\|\in\|_h \le \|\phi\|_h \equiv \sqrt{a(\phi,\phi)} = \sqrt{\sum_{K=1}^{N} a_K(\phi,\phi)} \quad a(u,v) = \int_K \left(\varepsilon \frac{du}{dx}\frac{dv}{dx} + 2uv \right) dx \tag{3}$$

The above error estimate is compared against the norm of the exact numerical error in order to assess its validity. Function φ is calculated by solving a local problem on each element by employing an element equilibration technique:

$$a_K(\phi_K, B_i) = -D(\phi_K, B_i) + B_i \varepsilon \frac{d\hat{y}}{dx}\bigg|_0^1 + B_i \left[\alpha_{K,K+1} \varepsilon \frac{d\hat{y}}{dx}\bigg|_{x_{K+1}}^{K+1\,elem} + \alpha_{K+1,K} \varepsilon \frac{d\hat{y}}{dx}\bigg|_{x_{K+1}}^{K\,elem} \right] -$$
$$B_i \left[\alpha_{K,K-1} \varepsilon \frac{d\hat{y}}{dx}\bigg|_{x_K}^{K-1\,elem} + \alpha_{K-1,K} \varepsilon \frac{d\hat{y}}{dx}\bigg|_{x_K}^{K\,elem} \right] \tag{4}$$

x_K and x_{K+1} are endpoints of the K'th element in the one-dimensional mesh, $\alpha_{K,K+1}$ $\alpha_{K+1.K}$ are appropriately chosen flux splitting constants [3,4] and B_i basis functions of higher order than those used in the solution of the actual problem (2).

In this fashion, upon obtaining a solution of (1) via (2) an a-posteriori error estimate is obtained. Next the mesh is subdivided in elements for which the indicator

$$\eta \equiv \frac{\|\phi\|_{h,K}}{\sqrt{\|\phi\|_{h,K}^2 + \|\hat{y}\|_{h,K}^2}} \tag{5}$$

falls below a certain predetermined value, e.g η=0.1, and fast convergence to the actual solution is obtained [5]. Extension to the two-dimensional case is underway in which case the finite element solution needs to be extended so that it accounts for irregular nodes that arise in the process of h-refinement of the mesh.

CONCLUSIONS AND GENERAL PERSPECTIVES

Upon extension to two-dimensional problems and proper coupling with a spectral methodology to account for variations in the third dimension, the above methodology is expected to provide increased accuracy in simulations of MHD flows at large Hartmann numbers, in which case strong Hartmann and side layers develop in the vicinity of walls in ducts carrying liquid metals. Especially in cases with curved boundaries where asymptotic solutions are difficult to develop this approach is a viable alternative for obtaining reliable numerical predictions of flows with large Hartmann.

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